APPLICATION NO. 09/826,118

TITLE OF INVENTION: ~~N~~ew Wavelet Multi-Resolution Waveforms

INVENTOR: Urbain Alfred von der Embse

BACKGROUND OF THE INVENTION

I. Field of the Invention

The present invention relates to CDMA (Code Division Multiple Access) cellular telephone and wireless data communications with data rates up to multiple T1 (1.544 Mbps) and higher (>100 Mbps), and to optical CDMA. Applications are mobile, point-to-point and satellite communication networks, data compression, pattern recognition, media image compression and processing, and radar More particularly, the present invention relates to Wavelet waveforms and filters for multi-resolution applications.

II. Description of the Related Art

Wavelets are waveforms of finite extent in time (t) and frequency (f) over the time-frequency (t-f) space, with multi-resolution, scaling, and translation properties. Wavelets over the analog and digital t-f spaces respectively are defined by equations (1) and (2).

<u>Continuous wavelet</u>

$$\psi_{a,b}(t) = |a|^{-1/2}\psi\left(\frac{t-b}{a}\right) \tag{1}$$

Digital or discrete wavelet

$$\psi_{a,b}(n) = |a|^{-1/2} \psi(\frac{n-b}{a}) \qquad (2)$$

where the two index parameters "a,b" are the Wavelet dilation and translation respectively or equivalently are the scale and shift, "t" is the continuous time variable, and "n" is the discrete sample variable for the digital representation assuming that the Wavelet is sampled at "T" second intervals. The ψ "mother" wavelet ψ is a real and symmetric localized function in the t-f space at baseband (dc, f=0). and generates the doubly indexed Wavelet ψ_{ab} where the scale factor "$|a|^{-1/2}$ keeps the Wavelet norm invariant under parameter changes "a,b", and norm is the square-root of the energy of the Wavelet response.

Wavelets in digital t-f space are an orthogonal basis with the choice of the parameters "a,b" equal to $a=2^{-p}$, $b=qM2^{p}$ where 'p,q" are the new scale and translation parameters and "M" is the spacing or repetition interval $T_s=MT$ of the Wavelets at the same scale "p". Wavelets at "p,q" are related to the mother Wavelet by the equation

$$\psi_{p,q}(n) \quad = 2^{-p/2}\psi(2^{-p}n - qM) \qquad (3)$$

Wavelets satisfy the orthogonality equation with a correlation value equal to "1" when suitably scaled:

$$\begin{aligned} \sum_n \psi_{p,q} \ \psi_{k,m} &= 1 \text{ iff both } p = k \text{ and } q = m \\ &= 0 \text{ otherwise} \end{aligned} \qquad (4)$$

FIG. 1 is an N point t-f space which is an N sample window of a uniform stream of digital samples at the rate 1/T

Hz (1/second) corresponding to a "T" second sample interval and illustrates a Wavelet representation or "tiling" with Wavelets that are designed analytically or by an iterated filter construction. The t-f space is covered or tiled by a set of Wavelet subspaces $\{W_p, p = 0,1,...,m-1\}$ 1 where $N=2^m$. Each Wavelet subspace W_p 1 at scale "p" consists of the set of Wavelet time translations $\{q = 0,1,..., N/2^{p+1}-1\}$ over this subspace. These Wavelet subspaces are mutually orthogonal and the Wavelets within each subspace are mutually orthogonal with respect to the time translates. This N-point t-f space extends over the time interval from 0 to (N-1)T 2 where T is the digital sampling interval, and over the frequency interval from 0 to (N-1) 3 in units of the normalized frequency fNT 4.

FIG. 2 is an iterated filter bank used to generate the Wavelets which cover the t-f space in FIG. 1. Each filter stage 5 consists of a high pass filter (HPF) and a low pass filter (LPF). Output 6 of the LPF is subsampled by 2 which is equivalent to decimation by 2. This t-f space space is an N-dimensional complex vector metric space $\mathbf{V}$ 7. At stage m in the iterated filter bank, the remaining t-f space $\mathbf{V}_m$ 8 is partitioned into $\mathbf{V}_{m+1}$ 9 and the Wavelet subspace W_{m+1} 10. Scaling functions and Wavelets at each stage of this filter bank satisfy the equations

$$\mathrm{LPF}_p: \quad \varphi(n) = 2^{-1/2}\sum_q h_q\,\varphi(2n-q) \quad \forall q \tag{5}$$

$$\mathrm{HPF}_p: \quad \psi(n) = 2^{-1/2}\sum_q g_q\,\varphi(n-q) \quad \forall q \tag{6}$$

where φ is the scaling function, ψ is the Wavelet, HPF_p coefficients are $\{h_q, \forall q\}$, LPF_p coefficients are $\{g_q, \forall q\}$, and the equations apply to the stages 0,1,...,m-1. The scaling

function $\varphi(n-q)$ in the Wavelet equation $\psi(n)$ can be changed to $\varphi(2n-q)$ to reflect a downsampling $\downarrow 2$ by the factor 2 if it is desirable to reduce the sample rate to the minimum required for the subspace and whereupon the $\downarrow 2$ would also appear in FIG. 2 as the final stage of signal processing for each of the Wavelet subspaces $W_0, W_1, \ldots, W_m$

HPF_p and LPF_p are quadrature mirror filters (QMF) with perfect reconstruction for our application. This means they cover the subspace V_p with flat responses over the subband frequency including the edges of the frequency subband, and the HPF_p coefficients are the frequency translated coefficients for the LPF_p: $\{g_q=(-1)^q h_q, \forall q\}$.

Wavelet design using iterated filter bank starts with the selection of the scaling functions. Starting with a primative scaling function, one can use the iterated filter construction given by equations (5) and (6) to derive successive approximations to a desired scaling function which has properties that have been designed into it by the selection of the filter coefficients $\{h_q, \forall q\}$ at each level of iteration. The Wavelets can be derived from these scaling functions using the iterated filter construction in FIG. 2 or scaling equations (5) and (6). Another use of the iterated filter construction is to design the scaling functions as Wavelets thereupon ending up with a larger set of Wavelets for multi-resolution analysis and synthesis.

SUMMARY OF THE INVENTION

This invention is a new and novel design of multi-resolution Wavelet waveforms in the frequency domain with the capability to be designed to meet specific performance

requirements and with a property which allows a single Wavelet design to be used for multiple scales.

Wavelets for linear applications to communications, radar, data compression, pattern recognition, and media image compression and processing, in this invention disclosure are assumed to use 1) LS metrics for the passband and stopband frequency requirements, 2) quadrature mirror filtering (QMF) LS error metric, 3) intersymbol interference (ISI) and adjacent channel interference (ACI) non-linear LS error metrics, 2) a subset of the frequency harmonics as the design coordinates and to convert the time domain specification of the LS metrics into the frequency harmonics when necessary, and to 3) design the Wavelet multi-resolution waveform for no excess bandwidth $\alpha=0$ where α is the excess bandwidth parameter. These metrics are weighed and summed to give a cost function J and iterative solution techniques are used to find the set of design harmonics which minimize J. These design harmonics generate the mother Wavelet and the mother Wavelet is scaled to generate the Wavelets at different scales specified by the dilation, time translation, and frequency translation parameters.

Non-linear applications of the Wavelets in this invention disclosure could require additional metrics and constraints not included in the linear waveform and filter applications examples and are within the scope of this invention disclosure. Examples given in this invention disclosure are the constant amplitude minimum-shift-keying MSK bandwidth efficient BEM Wavelet waveforms for communications and the zero-sidelobe synthetic aperature radar SAR Wavelet waveforms for radar.

Scope of this invention disclosure covers decisioning rules and metrics other than LS techniques, media (spatial dimension)-f space, pattern recognition features-f space, satellite and

cellular t-f-beam space, and optical and laser transmission time-wavelength space.

BRIEF DESCRIPTION OF THE DRAWINGS AND THE PERFORMANCE DATA

The above-mentioned and other features, objects, design algorithms, and performance advantages of the present invention will become more apparent from the detailed description set forth below when taken in conjunction with the drawings and performance data wherein like reference characters and numerals denote like elements, and in which:

FIG. 1 is an N-point t-f space extending over the time interval (0, (N-1)T] and the frequency interval (0, (N-1)/NT], which is tiled or covered by a set of orthonormal Wavelets at the scales $p=0,1,\ldots,m-1$,

FIG. 2 is a Wavelet iterated filter bank used to generate the set of Wavelets which tile or cover the t-f space in FIG. 1,

FIG. 3 is the power spectral density (PSD) template for the the stopband and passband used to construct the LS error metrics,

FIG. 4 is the flow diagram of the LS error metrics leading to the construction of the LS error cost function J equal to the sum of the weighted LS error metrics and which is minimized through selection of the design harmonics to find the optimal mother Wavelet,

FIG. 5 is the Matlab 5.0 code for the LS recursive solution algorithm used to design the Wavelet waveform in FIG. 6 using an

eigenvalue technique, lists the Wavelet time response and the design harmonic values, and gives the code for scaling the design to a multi-resolution application specified by dilation (scale), time translation, and frequency translation parameters.

FIG. 6 plots the dc PSD in dB units for the frequency response of the mother Wavelet at dc and the square-root raised-cosine (sq-rt r-c) waveforms with excess bandwidth parameter α= 0.22, 0.40 vs. the normalized frequency in units of symbol rate $1/T_s$.

FIG. 7 plots the dc PSD in dB units for the frequency response of the mother Wavelet at dc and an optimized Gaussian minimum shift keying (GMSK) waveform vs. the normalized frequency fT_s. The Wavelet is designed using the LS algorithms with additional constraints that reflect the application to the minimum shift keying constant amplitude bandwidth efficient modulation (BEM) waveform,

FIG. 8 plots the amplitude of the dc radar ambiguity function for a synthetic array radar (SAR) application for the mother Wavelet at dc and an optimized unweighted chirp waveform vs. the normalized frequency fT_p and normalized time t/T_c where the pulse time T_p and chip time T_c are both equal to the symbol interval T_s. The Wavelet is designed using the LS algorithms with the constraint that the SAR waveform have no sidelobes.

DISCLOSURE OF THE INVENTION

Wavelet multi-resolution waveforms incorporate a frequency translation, are complex, are designed in frequency harmonic coordinates, a single mother Wavelet design is used for all scales, and in the t-f space are realized as finite impulse response (FIR) filters and waveforms.

1. Wavelet multi-resolution waveform

Multi-resolution Wavelets at the scales "p,q,r" are related to the mother Wavelet at dc by the equation

$$\psi_{p,q,r}(n) = 2^{-p/2}\psi(2^{-p}n - qM)\,\exp[-j2\pi f_c(p,r)(2^{-p}n)(2^{p}T)] \qquad (7)$$

where $f_c(p,r)$ is the center frequency of the frequency translated mother Wavelet at dc as a function of the scale "p" and frequency index "r" and the phase factor $(2^{p}T)$ reflects the observation that the sampling interval "T" is increased to "$2^{p}T$" under the scale change from "p=0" to "p=p" The purpose of the frequency index "r" is to identify the center frequencies of the Wavelets at the scale "p" and translation "q" in the t-f space. These Wavelets satisfy the complex orthogonality equations

$$\begin{aligned}\sum_n \psi_{p,q,r}\psi^*{}_{k,m,v} &= 1 \quad \text{iff } p=k,\ q=m,\ r=v \qquad (8)\\ &= 0 \quad \text{otherwise}\end{aligned}$$

where "*" is conjugation. Wavelet basis vectors for the metric V=t-f space consist of a subset of the admissible set of scaled and translated Wavelets $\{\psi_{p,q,r}, \forall p,q,r\}$ derived from the mother

Wavelet ψ as per equation (7). An admissible Wavelet waveform is any combination that covers V=t-f space. The discrete Fourier transform (DFT) harmonic coefficients $\{\psi_k\}$ and the continuous Fourier transform spectrum $\psi(\omega)$ are defined by the equations

$$\psi(\omega) = \sum_n \psi(n)\, e^{-i\omega n} \quad \text{Fourier Transform} \qquad (9)$$

$$\psi_k = \sum_n \psi(n)\, W_{N'}^{-kn} \quad \text{DFT harmonic coefficients}$$

where

$$\begin{aligned}\psi(\omega) &= (1/N')\sum_k \psi_k \sum_n e^{i(2\pi k/N' - \omega)n} \\ &= \sum_k \psi_k \sin((\omega/2 - \pi k/N')N') / N' \sin(\omega/2 - \pi k/N') \\ &= \sum_k \psi_k \left[\text{Harmonic interpolation for "k"}\right]\end{aligned}$$

$W_{N'}^{kn} = e^{i2\pi kn/N'}$

$\{k\}$ = *DFT frequency or harmonic coefficients such that* $f_k = k/N'T$ *where* f_k *is the harmonic frequency corresponding to "k"*

N' = *length of* $\psi(n)$

Orthonality and no excess bandwidth $\alpha=0$ are properties which are asymptoticly approached by our Wavelets to within design accuracies inherent in communications and radar. The Wavelets derived from the mother Wavelet are designed to be orthogonal over both time translates "MT" and frequency translates "1/LT" which respectively correspond to the Wavelet symbol spacing T_s=MT and the adjacent channel spacing $1/T_s$. This means the orthogonal spacing of the waveforms in V are at the time-frequency increments (MT, 1/LT) = (T_s. $1/T_s$). In the interests of constructing our multi-resolution Wavelets to cover V it will be convenient to assume that M,L are powers of 2. We need the following definitions for the parameters and coordinates.

Parameters and Coordinates (10)

N' = Length of ψ which is an even function about the center and which spans an odd number of points or samples

= $ML+1$ where M, L are assumed to be even functions for convenience of this analysis

= Number of points of ψ

M = Sampling interval for ψ

= Spacing of ψ for orthogonality

L = Length of ψ in units of the sample interval M

= Stretching of ψ over L sample intervals

n $= n_0 + n_1 M$

= partitioning into an index n_0 over the sample length $n_0 = 0,1,\ldots,M-1$ and an index $n_1 = 0,1\ldots,L-1$ over the sample intervals

k $= k_0 + k_1 L$

= partitioning into an index k_0 over the harmonic frequencies $k_0 = 0,1,\ldots,L-1$ corresponding to the stretching and an index $k_1 = 0,1,\ldots,M-1$ over the harmonic frequencies corresponding to the admissible frequency slots for ψ

The harmonic design coordinates are the subset of harmonics which span the spectral containment. For the example algorithm development in this invention disclosure it is assumed the waveforms are spectrally contained in the frequency interval 1/MT corresponding to the filter spacing which means the L harmonics $\{k_0=0,1,\ldots,L-1\}$ covering this spacing are the design harmonics. For other applications as will be demonstrated later, the spectral containment is spread out and one must increase the

subset of design harmonics. The DFT equations for the mother Wavelet in (9) when rewritten in terms of the L harmonic design coordinates $\{\psi_{k_0}, \forall k_0\}$ become:

DFT equations for the mother Wavelet at dc (11)

$$\psi_{k_0} = \sum_{n} \psi(n)\ W_{N'}^{-k_0 n} \qquad \text{design harmonics}$$

$$\psi(n) = (1/N') \sum_{k_0} \psi_{k_0}\ W_{N'}^{k_0 n} \qquad \text{Wavelet}$$

2. Qrthogonal basis for t-f space

The Wavelet orthogonality property and tiling of the t-f space with $\alpha=0$ will be proven using the theorems of Karhunen-Loeve and Mercer. We start by considering the expansion of a random complex sequence $\{z(n), \forall n\}$ in a series of Wavelet coordinates consisting of time translates of the mother Wavelet ψ. The sequence $\{z(n), \forall n\}$ is a zero-mean stationary random process which is orthonormal over the sample interval "M" and has a frequency spectrum which is flat and extends over the frequency range 1/MT which is centered at baseband corresponding to a zero frequency. This means the $\{z(n), \forall n\}$ cover the subspace of V corresponding to the scale of our mother Wavelet ψ and its time translates $\{\psi(n-qM) = \psi_q(n), \forall q\}$. In addition, the V is now considered to be extended over a time interval which is relatively large compared to the N-dimensional t-f space in FIG. 1,2 to avoid end-effects on the analysis. We start by approximating the sequence $\{z(n), \forall n\}$ by the $\{\hat{z}(n), \forall n\})$ as per the following equations.

$$\begin{aligned}\hat{\mathbf{z}}(\mathbf{n}) &= \sum_{q} \mathbf{Z}_q\, \psi(n - qM) \\ &= \sum_{q} \mathbf{Z}_q\, \psi(\mathbf{n}_0 + (\mathbf{n}_1 - \mathbf{q})\mathbf{M})\end{aligned}$$

where the complex coefficients $\{Z_q\}$ are derived from the original sequence (12)

$$\begin{aligned} Z_q &= \sum_{n} z(n)\, \psi^*(n - qM) \\ &= \sum_{n} z(n)\, \psi_q(\mathbf{n})^* \end{aligned}$$

wherein the the coefficients $\{Z_q, \forall q\}$ are orthonormal as proven by the equations:

$$\begin{aligned}\mathbf{Z}_q\mathbf{Z}_{q'}^* &= \sum_{\Delta n} \mathbf{z}(\Delta\mathbf{n} - \mathbf{qM})\mathbf{z}^*(\Delta\mathbf{n} - \mathbf{q'M})\psi_q(\Delta\mathbf{n})\psi_{q'}^*(\Delta\mathbf{n}) \qquad (13) \\ &= \delta_{qq'} \sum_{n} |\psi_q(\mathbf{n})|^2 \quad \text{since the sequence } \{z\} \text{ is orthonormal} \\ &\qquad\qquad\qquad\qquad \text{over the sampling interval "M"} \\ &= \delta_{qq'} \qquad\qquad\qquad \text{with normalization of the energy of } \psi \end{aligned}$$

Equations (12) and (13) together prove the Karhunen-Loeve theorem which proves the following equation for the accuracy in approximating the stochastic sequence $\{z(n), \forall n\}$ by $\{\hat{z}(n), \forall n\}$. This accuracy is expressed by the expected "E(o)" squared error "(o)" in this approximation:

$$E\{|z(n) - \hat{z}(n)|^2\} = 1 - \sum_{n_1} |\psi(n_0 + n_1 M)|^2 \qquad (14)$$

We need to prove that the right hand side of this equation is zero which then proves that the approximating sequence is equal to the original sequence in the mean-square sense. In turn this proves that the new waveform coordinates $\{\psi_q, \forall q\}$ are an orthogonal basis for the original sequence $\{z(n), \forall n\}$ with $\alpha=0$ which is our goal.

The right hand side of equation (14) when set equal to zero expresses Mercer's theorem so our goal is to prove Mercer's theorem. To do this we use the DFT of ψ in equation (11) and the coordinates in (10) to evaluate the right hand side of equation (14). We find

$$\begin{aligned}
1 - \sum_{n_1} \left|\psi(n + n_1 M)\right|^2 &= 1 - \sum_{k_0}\sum_{k_0'} \Psi_{k_0}\Psi_{k_0'} W_{N'}^{\Delta k_0 n_0}\,\Gamma \\
\textit{where} \quad \Gamma &= (1/L)\sum_{n_1} W_L^{\Delta k_0 n_1} \\
&= \sin(\pi\,\Delta k_0)/L\sin(\pi\,\Delta k_0/L) \\
&= 1 \ \text{ for } \Delta k_0 = 0 \qquad (15) \\
&= 0 \ \text{ otherwise}
\end{aligned}$$

This proves that

$$1 - \sum_{n_1} \left|\psi(n + n_1 M)\right|^2 = 0 \quad \forall n_0$$

which proves that equation (18) reduces to

$$E\left\{\left|z(n) - \hat{z}(n)\right|^2\right\} = 0 \qquad (16)$$

which as per the above proves that the set of Wavelets is an orthogonal basis and from the construction that $\alpha=0$. This proof easily generalizes to the multi-resolution Wavelets at all of the scales {p} and time translates {q} and frequency translates {r}, and to the expansion of the harmonic design coordinates as required by the application.

3. Single Wavelet design for multi-scales

We will demonstrate that a single Wavelet design can be used for all scales by 1) implementing the multi-scale transformations of the mother Wavelet with the Fourier domain

design harmonics, 2) by proving that the mother Wavelet design remains invariant under scale changes.

First consider the multi-scale transformation which derives the Wavelets at the scale and shift parameters "p,q" for the mother Wavelet at scale "p=0" and centered at the origin "q=0". We begin by extending the parameters and coordinates in equation (10), to include both scaling and subsampling or decimation in a form that is equivalent to the iterated filter bank construction which is used to derive the Wavelet using the filter scaling functions. Starting with the coordinates at scale "p=0" the parameters and coordinates at scale "p=p" are given by the equations:

P=0 parameters and coordinates (17)

$n = n_0 + n_1 M$

where

$n_0 = a_0 + a_1 2 + \ldots + a_{m-1} 2^{m-1}$

$= M$ points

$M = 2^m$

$n_1 = b_0 + b_1 2 + \ldots$

$=$ points spaced at M sample intervals

p=p parameters and coordinates

$2^{-p} n = n$ scaled by "2^{-p}"

$= n(\downarrow 2^p)$, n decimated by $2^p{:}1$

$= n_0(p) + n_1 M$

where

$n_0(p) = n_0$ scaled by "2^{-p}"

$= n_0(\downarrow 2^p)$

$= a_p + a_{p-1}2 + \ldots + a_{m-p-1}2^{m-p-1}$

$= M$ points spaced at 2^p sample intervals

Combining these equations with the analytical formulation in (7) and the Fourier domain representation in (11) enables the

Wavelets for "p,q,r" to be written as a function of the Fourier domain harmonic design coordinates:

$$\psi_{p,q,r} = (2^{-p/2} / \mathbf{N'}) \sum_{k_0} \psi_{k_0} W_{N'}^{k_0(n(p)-qM)} \, e^{i2\pi f_c(p,r)n(p)2^pT} \tag{18}$$

for all admissible scale, translation, and frequency index parameters, "p,q,r".

Next we need to demonstrate that the frequency domain design in (11) remains invariant for all parameter changes and in particular for all scale changes. This multi-scale property expresses the accordion behavior of the design in that the Wavelets at different scales are simply the stretched and compressed versions of the mother Wavelet with the appropriate frequency translation indices. This multi-scale invariancy means that the design for a M=16 channel filter bank remains the same for M=100 or M=10,000 channel filter banks, when the overlap L and the performance goals remain constant. To demonstrate this invariant property across scales, we consider the multi-resolution Wavelet at scale "p" with the other parameters set equal to zero for convenience "q=0, r=0" and without loss of generality. The Fourier domain frequency response ψ(f) can be evaluated starting with the original formulation in equation (7):

DFT at "p, q=0, r=0" (19)

$$\psi_p(\mathbf{f}) = (1/\mathrm{N'}) \sum_{k_0} \psi_{k_0} \sum_{n(p)} \mathbf{W}_{N'}^{-(fN'2^pT - k_0)n(p)}$$

$$= \sum_{k_0} \Psi_{k_0} \left[\frac{\sin(\pi \ (fN'2^pT - k_0))}{N'\sin(\pi \ (fN'2^pT - k_0)/N')} \right]$$

$$= \sum_{k_0} \Psi_{k_0} \ [\text{Harmonic interpolation for "}k_0\text{"}]$$

This only differs from the harmonic representation in equation (9) in the restriction of the design coordinates to the subset of harmonic coefficients $\{\psi_{k_o}, \forall k_0\}$ and the stretching of the time interval to "2^pT" corresponding to the scale "p=p". The harmonic interpolation functions are observed to remain invariant over scale changes upon observing that the frequency scales as "$f \sim 1/2^pT$" which means the (frequency*time) product remains invariant with scale changes as per the fundamental property of the Wavelets. This means the Wavelet design is an invariant across the Wavelet scales which means we only need a single design for all scales of interest.

4. Introduction to LS design

Design of the Wavelets for a uniform polyphase filter bank is described in detail in this invention disclosure. The t-f space is spanned by the uniform polyphase filter bank consisting of M channels at the frequency spacing $f_w=1/MT$ where T is the digital sampling interval, and the Wavelet filter finite impulse response (FIR) time response is stretched over L sampling time intervals T_s. This polyphase filter bank is ideally decimated which means the filter ouput sample rate $1/T_s$ is equal to the channel-to-channel spacing $T_s=MT$ which is equivalent to stating that there is no excess bandwidth $\alpha=0$. Our design for this topology is immediately applicable to an arbitrary set of multi-resolution Wavelet filters through the scaling equation (18) which gives the design of our Wavelet at arbitrary scales in terms of our design of the mother Wavelet at dc.

LS design algorithms are the eigenvalue and the gradient search implementation of an LS algorithm and which respectively can be reduced to algorithms that are equivalent to the original

eigenvalue and Remez-exchange Wavelet design algorithms for application to a uniform filter bank. Design of the mother Wavelet for this polyphase filter bank uses 5 LS error metrics consisting of the 2 prior art passband and stopand metrics which are upgraded for our mother Wavelet design, and the 3 new metrics consisting of the ISI, ACI, and QMF, and solve the LS minimization problem using as design coordinates the subset of harmonic coordinates.

Frequency domain design coordinates are related to the Wavelet time domain digital samples or coordinates as follows.

Mapping of frequency to time (20)

Time domain design coordinates $\{\psi(n), \forall \mathrm{n}\}$ are real and symmetric and can be represented by the reduced set $\{h_t(n), n=0,1,\ldots,ML/2\}$

$$
\begin{aligned}
h_t(n) &= \psi(0) \quad \text{for } n=0 \\
&= 2\psi(n) \quad \text{for } n=1,2,\ldots,ML/2 \\
&= \text{time domain design coordinates}
\end{aligned}
$$

Frequency domain harmonic design coordinates $\{\psi_{k_o}, \forall k_0\}$ are real and symmetric and can be represented by the reduced set $\{h_f(k),\ k=0,1,,\ldots,L-1\}$

$$
\begin{aligned}
h_f(k) &= \psi_{k_0} \quad \text{for } k=k_0=0 \\
&= 2\psi_{k_0} \quad \text{for } k=k_0=1,2,\ldots,L-1 \\
&= \text{frequency domain design coordinates}
\end{aligned}
$$

Mapping of the frequency coordinates $\{h_f(k),\ k=0,1,\ldots,L-1\}$ into the time coordinates $\{h_t(n),\ n=0,1,\ldots,ML/2\}$ is defined by the matrix transformation

$$h_t = B \quad h_f$$

where

$h_f = (h_f(0), \ldots, h_f(ML/2))^t$ transpose of column vector

$h_t = (h_t(0), \ldots, h_t(ML/2))^t$ transpose of column vector

$B = (ML/2 + 1) \times L$ matrix

$= [\ B_{kn}\ ]$ matrix of row k and column n elements B_{kn}

$B_{kn} = 1\ /ML$ for n=1

$= 2 \cos(2\pi kn/\ ML)$ otherwise

wherein the N'=ML+1 has been replaced by ML since a single end point has been added to the the FIR to make it symmetrical for ease of implementation for the example Wavelet being considered with a sample at the mid-point which makes the number of samples N' an odd number.

5. Passband and stopband LS error metrics

Passband and stopband LS error metrics and cost functions for FIR waveforms and filtering are derived with the aid of FIG. 3 which defines the power spectral density (PSD) parameters of interest for the passband and stopband of the PSD $\Psi(\omega)$ for communications applications. Requirements for radar applications include these listed for communications. Referring to FIG. 3 the passband 11 of the Wavelet PSD is centered at dc (f=0) since we are designing the or mother Wavelet, and extends over the frequency range ω_p extending from $-\omega_p/2$ to $+\omega_p/2$ 12 in units of the radian frequency variable $\omega=2\pi fT$ 13 where T is the digital sampling interval defined in FIG. 1. The frequency space extends over the range of f=-1/2T to f=+1/2T which is the frequency range in FIG 1 translated by -1/2T so that the mother Wavelet is at the center of the frequency band. Quality of the PSD over the passband is expressed by the passband ripple 14. Stopband 15 starts at the edge 16 of the passbands of the adjacent channels $+/-\omega_a/2$ 16 and extends to the edge of the frequency band $\omega=+/-\pi$ 17 respectively. Stopband attenuation

18 at $+/-\omega_a/2$ measures the PSD isolation between the edge of the passband for the mother Wavelet and the start of the passband for the adjacent channels centered at $+/-\omega_s$ 19. Rolloff 20 of the stopband is required to mitigate the spillover of the channels other than the adjacent channels, onto the dc channel. Deadband or transition band 21 is the interval between the passbands of contiguous channels, and is illustrated in FIG. 3 by the interval from $\omega_p/2$ to $\omega_a/2$ beween the dc channel and adjacent channel at ω_a. Wavelet sample rate ω_s 22 is the Wavelet repetition rate. For the LS example algorithms, the Wavelet sample rate is equal to the channel-to-channel spacing for zero excess bandwidth. Therefore, $1/T_s= \omega_s/2\pi T=1/MT$ which can be solved to give $\omega_s=2\pi/M$ for the radian frequency sampling rate of the filter bank which is identical to the Wavelet repetition rate.

We start by rewriting the DFT equations for the dc Wavelet in (11) as a function of the $\{h_f(k),\ k=0,1,\ldots,L-1\}$

$$\begin{aligned} \psi(\omega) &= \sum_n \psi(\mathrm{n})\cos(\mathrm{n}\omega) \\ &= \mathrm{c}^t\,\mathrm{B}\,\mathrm{h}_f \quad \text{using (21) and the definition of the vector "c"} \qquad (21) \\ \mathrm{c} &= (1,\cos(\omega),\ldots,\cos((\mathrm{ML}/2)\,\omega))^t \quad \text{transpose of column vector} \end{aligned}$$

which is equivalent to the equation for $\psi(\omega)$ in (9) expressed as a linear function of the $\{h_f(k),\ k=0,1,\ldots,L-1\}$. However, this functional form is more convenient to analyze. A ideal "c" vector "c_r" will be introduced for the passband and the stopband in FIG. 3, in order to identify the error residual $\delta\psi(\omega)$ at the frequency "ω" in meeting the ideal passband and stopband requirements. The ideal PSD is flat and equal to "1" for the passband, and equal to "0" for the stopband. We find

Error residuals for passband and stopband (22)

$c_r \;=\; (1,1,\ldots,1)^t$ passband ideal "c"

$\;\;\;\;=\; (0,0,\ldots,0)^t$ stopband ideal "c"

$\delta c \;=\; c_r - c$ error vector

$$\delta\psi(\omega) = \delta c^t \, B \, h_f$$

$$= \text{Residual error in meeting the ideal spectrum at "}\omega\text{"}$$

The LS metric for the passband and stopband can now be constructed as follows for the eigenvalue and the LS optimization (or equivalently, the LS algorithm) design algorithms.

Passband and stopband metrics (23)

$$J(\text{band}) = \frac{1}{band}\int_{\text{band}} \left|\delta\psi(\omega)\right|^2 \, d\omega \qquad \text{Eigenvalue}$$

$$= h_f^t \, R \;\, h_f \qquad \text{Eigenvalue}$$

$$= \left\|\delta\psi\right\| \qquad \text{LS}$$

where

$\text{band} \;=\; [0,\; \omega_p\,)$ passband

$\;\;\;\;\;\;=\; (\omega_s\;,\;\pi]$ stopband

$$R \;=\; \frac{1}{band}\int_{\text{band}} (B^t \, \delta c \, \delta c^t \, B)\, d\omega$$

$= L \times L$ matrix

$\delta\psi \;=\; (\;\delta\psi(\omega_1)\;,\ldots,\;\delta\psi(\omega_u)\;)^t$

= vector of error residuals at the frequencies $\omega_1\;,\;\ldots,\;\omega_u$ across the band

$\left\|(o)\right\|$ = norm or length of the vector (o) and which includes a cost function for the errors of the individual components

where it is observed that the eigenvalue approach requires that the LS metrics be given as quadratic forms in the design

coordinates $\{h_f(k),\ k=0,1,\ldots,L-1\}$ whereas with the LS approach it is sufficient to give the LS metrics as vector norms with imbedded cost functions or an equivalent formulation.

6. QMF LS error metric

Wavelet quadrature mirror filtering (QMF) is a scaling property that supports iterative filter construction of Wavelets, tiling of the t-f space, and perfect reconstruction for polyphase filtering. QMF metrics express the requirements on the deadband that the PSD's from the contiguous channels in FIG. 3 add to unity across the deadband $[\omega_p,\ \omega_s]$ in order that the filters be QMF filters. By suitable modification of the error vector δc, the previous construction of the passband and stopband metrics can be modified to apply to the deadband. We find

Deadband metrics (24)

$$J(\text{deadband}) = \frac{1}{deadband}\int_{\text{deadband}} |\delta\psi(\omega)|^2\, d\omega \qquad \text{Eigenvalue}$$

(24) continued

$$= h_f^t R\ h_f \qquad \text{Eigenvalue}$$

$$= \|\delta\psi\| \qquad \text{LS}$$

where

$$\delta c = c_r - c(\omega) - c(\pi/M - \omega)$$

where $c(\omega)=c$ as defined in (22) and (23), and $c(\pi/M - \omega) = c$ at the offset frequency "$\pi/M - \omega$" corresponding to the overlap of the contiguous filters over the deadband.

7. ISI and ACI LS orthogonal error metrics

Wavelet orthogonality metrics measure how close we are able to designing the set of Wavelets to be orthogonal over the t-f space, with the closeness given by the ISI and the ACI. ISI is the non-orthogonal error between channel output samples separated by multiples of the sampling interval 1/MT seconds where T is the sample time and M is the interval of contiguous samples. ACI is the non-orthogonal error between between channel output samples within a channel and the samples in adjacent channels at the same sample time and at sample times separated by multiples of the sample interval. When observed as noise contributions within each sample in a given channel, the ISI is the noise contribution due to the other received Wavelets at the different timing offsets corresponding to multiples of the sampling interval. Likewise, the ACI is the noise contribution due to the other Wavelets in adjacent channels at the same sampling time and at multiples of the sampling interval.

ISI and ACI errors are fundamentally caused by different mechanisms and therefore have separate metrics and weights to specify their relative importance to the overall sum of the LS metrics. ISI is a measure of the non-orthogonal between the stream of Wavelets within a channel as per the construction in FIG. 3. On the other hand, ACI is a measure of the non-orthogonal between the Wavelet within a channel and the other Wavelets in adjacent channels. This means the stopband performance metric has a significant impact on the ACI due to the sharp rolloff in frequency of the adjacent channel, and the ACI metric is then a measure of the residual non-orthogonal due to the inability of the stopband rolloff in frequency from completely eliminating the ACI errors.

We assume the received Wavelet is identical to the filter Wavelets and is transmitted at the filter output sample intervals equal to 1/MT seconds. The second assumption means we are assuming the receiver is synchronized with the received signal. Since there is no information lost by sampling asynchronously with the received Wavelet, we are free to make this synchronization assumption without loss of generality. ISI metrics are derived in the following set of equations.

ISI metrics (25)

Mapping of h_f into ψ

$\psi = (\psi(-ML/2), \ldots, \psi(ML/2))^t$ transpose of column vector

$\quad = H\ h_f$

$H = (ML+1) \times L$ matrix of elements H_{kn}

$H_{kn} = 1$ for $n=1$

$\quad = 0.5 \cos(2\pi kn/ML+1)$ otherwise

Offset matrix A

$A = L \times (ML+1)$ matrix of elements A_{kn}

$A_{kn} = 0$ for $k=1$

$$= \begin{array}{c} \left[\begin{array}{ccccccc} 0 & 0 & \ldots & \psi(-ML/2) & \ldots & \psi((L-k)M+1) \end{array}\right] \\ \begin{array}{ccccccc} 1 & 2 & & kM & & LM+1 \end{array} \end{array}$$

(25) continued

ISI error vector δE

$\delta E = L \times 1$ column vector

$\quad = A\ H\ h_f$

ISI metric

$J(ISI) = \delta E^t\ \delta E$ Eigenvalue

$\quad =$ Non-linear quadratic function of h_f

$\quad = \|\delta E\|$ LS

ACI metrics are derived using the ISI metric equations with the following modifications.

ACI metrics (26)

Mapping of h_f into ψ is the same as developed for ISI

Offset matrix A elements are changed as follows to apply to Channel 1:

$$
\begin{aligned}
A_{kn} &= 0 \qquad \text{for } k=1 \\
&= \begin{array}{cccccccc} [0 & 0 & \ldots & \psi(-ML/2)\,W_M^0 & \ldots & \psi((L-k)M+1) & W_M^{(L-k)}] \\ 1 & 2 & & kM+1 & & LM+1 & \end{array}
\end{aligned}
$$

which means the ACI error vector δE is

$$
\begin{aligned}
\delta E &= L \times 1 \quad \text{column vector} \\
&= A\,H\,h_f
\end{aligned}
$$

ACI metric for the two contiguous channels

$$
\begin{aligned}
J(ISI) &= 2\,\delta E^t\,\delta E \qquad \text{Eigenvalue} \\
&= \text{Non-linear quadratic function of } h_f \\
&= 2\,\|\delta E\| \qquad \text{LS}
\end{aligned}
$$

where the factor "2" takes into account there are two contiguous channels or one on either side of the reference channel 0 in FIG. 3. Because of the fast rolloff of the frequency spectrum the addition of more channels into the ACI metric is not considered necessary, although the functional form of the ACI metric in (26) allows an obvious extension to any number of adjacent channels which could contribute to the ACI.

8. LS error cost metric J

Cost metric or function J for the LS algorithms is the weighted sum of the LS metrics derived in (23), (24), (25), (26). The LS algorithms minimize J by selecting the optimal set of frequency coordinates $\{h_f(k), \forall k\}$ for the selected set of

parameters used to specify the characteristics of the dc Wavelet, frequency design coordinates, LS metrics, and weights. Cost function and design optimization goals are defined by the equations:

Cost function metric J (27)

$$J = \sum_{x} w(metrics)\, J(\mathrm{metrics})$$

$= $ weighted sum of the LS metrics J(metrics)

where

metrics = passband, stopband, deadband, ISI, ACI

$\{w(\mathrm{metrics}), \forall metrics\}$ = set of weights

$\sum_{x} w(metrics) = 1$ normalization

Optimization goal

Goal: minimize J with respect to the selection of the $\{h_f(k), \forall k\}$

9. LS error design

Two classes of LS optimization algorithms considered in this invention disclosure as examples for solving equations (27) are an eigenvalue and a gradient search implementation of an LS algorithm. The eigenvalue algorithm uses a non-linear quadratic formulations of the LS metrics and the LS algorithm uses the norm formulations for the LS metrics. LS optimization is a class of search algorithms which find the minimum LS error cost function J.

FIG. 4 is a summary of the LS metrics and the construction of the cost function J. Design parameters 23 are the input and

output design parameters. Input parameters are the number of polyphase channels M or equivalently the number of digital samples at spacing T over the symbol interval T_s=MT, the length of the FIR time response for the Wavelet in units of L which are the number of digital samples per Wavelet repetition interval T_s so that the total number of digital samples for the symmetric FIR time response is equal to ML+1, number of DFT samples per FIR length n_fft for implementation of the LS algorithms, passband radian frequency ω_p, stopband radian frequency ω_a, Wavelet repetition rate in radian frequency ω_s, selection of the set of design coordinates $\{h_f\}$ to be used in the optimization, and the metric weights {w(metrics)}. Output parameters are the set of harmonic design coordinates $\{h_f\}$ that minimize J. Band metrics 24 are the passband, stopband, and deadband metrics defined in equations (23), (23), (24) respectively. Interference metrics 25 are the ISI and ACI metrics defined in equations (25) and (26) respectively. LS cost function J 26 is the weighted linear sum of the metrics defined for the band 24 and the interference 25 as defined in equation (27).

10. Matlab 5.0 code for eigenvalue design

FIG. 5A,5B,. . .,5N,5O are the Matlab 5.0 code for the LS recursive eigenvalue solution algorithm used to design the Wavelet Wavelet in FIG.6. Included are a listing of the frequency domain design harmonics, time response, and data plotting software used in the optimization, and an example of the scaling of this mother Wavelet into a Wavelet with new scaling (dilation), time translation, frequency translation parameters, and the functions used in the code. The Matlab code uses an iterative approach to find the optimal frequency domain eigenvector solution to the quadratic form for the LS error whose error matrix is the weighed sum of the error matrices for the

stopband, passband, deadband metrics and the error matrices for the ISI and ACI metrics which are functions of the eigenvectors. For each iteration the ISI and ACI error matrices are constructed with the frequency domain eigenvectors from the previous iteration. The iteration finds the new frequency domain eigenvectors which are then used to update the ISI and ACI metrics for the next iteration. This iteration is repeated until there is convergence. The Matlab code allows the operator to update the metric weights as well as the scenario parameters to find the optimal choice for the performance goals.

FIG 5A Matlab code starts with the selection of the design parameters in 29 Step 1. Step 1.1 lists the scenario parameters which are M=16 digital samples per Wavelet ψ sample interval, L=16 nominal length of Wavelet ψ in units of M, N=ML+1 Wavelet ψ length which is represented by N' in equations (10), fs=1 is the normalized channel spacing equal to the Wavelet ψ sample rate, fp=0.8864 is the passband frequency interval, n_f=16 is the number of design frequency harmonics, n_fft=1024 is the number of digital samples used for calculation of the frequency spectrum centered at 0 frequency. ebno=E_b/N_o=6.0 dB is the design value of the energy per bit E_b to noise power density N_0 ratio, and x_imbal_aci=6.0 dB is the assumed channel-to-channel power imbalance used in the calculation of the ACI errors. Step 1.2 derives the software parameters from the design parameters in Step 1.1. Step 1.3 lists the optimization parameters which are the number of iterations n_iteration = 10, and the metric weights w(metric) from equation (27) and in FIG. 4 equal to w(pass)= alpha_1=1.e-2 for passband, w(stop)=alpha_2=0.80 for stopband, w(ISI)=alpha_3=2.e-3 for ISI, w(ACI)=alpha_4=0.5 for ACI, and w(dead)=alpha_5=0 for deadband. The number of iterations was selected to provide a convergent solution for the weighted LS error metrics w(metric)J(metric) in FIG. 5K 38 Step 10 figure(2) and for their sum J in figure(1). Weight values were selected to

optimize the Wavelet ψ filter performance in FIG. 6 and in figure(3) in FIG. 5L 39 Step 11, Wavelet ψ ripple, ISI, ACI signal-to-noise SNR power ratio losses in figure(4), and Wavelet ψ time response in figure(5).

FIG. 5B 30 Step 2 are the initialization calculations prior to the start of the iteration loop. Step 2.1 calculates the one-sided Wavelet ψ length parameter m=128. Step 2.2 calculates the matrix transformation bw_matrix which maps the ψ frequency domain design eigenvectors into the one-sided ψ time response. Step 2.3 refers to the function pmn in FIG. 5P,5Q 41 in Step 13.1 which computes the LS error matrix which is the sampled data integral $\Sigma[\delta c \bullet \delta c']d\omega$ over the band in equation (23) for the the passband and stopband metrics J(pass),J(stop). The function pmn_d computes the corresponding integral over the deadband in equation (24) and is not listed since it was not used for FIG. 6. Step 2.4 constructs a structural matrix c_matrix used in the ISI and ACI calculations. Step 2.5 constructs the sample rate templates for the passband upper edge, sample rate marker, and passband lower edge for frequency performance evaluation.

FIG. 5C 31 Step 3 uses the function pmn to compute the LS error matrices passband and stopband in Step 3.1, 3.2. Deadband LS error matrix deadband is set equal to a null matrix in Step 3.3. Step 3.4 computes the matrix p_total equal to the weighted sum of these matrices. Step 3.5 maps this LS error matrix p_total in the time domain into the LS error matrix pw_t in the frequency domain using the matrix transformation bw_matrix.

FIG. 5C,5D 32 Step 4 initializes matrices used in the iteration loop and starts the iteration loop for the iterations i_iteration=1,…,10. Step 4.1 calculates the eigenvalue and eigenvector which minimize the cost function J in equations (27) and FIG. 4 whose LS error matrix is pw_t. Step 4.2 uses the

bw_matrix to map the eigenvector in frequency into the Wavelet time response ψ=hn and stores the eigenvector as the set of Wavelet ψ design harmonics ψ_k=hw_eig(k) for k=0,1,…,15 where ψ_k has been defined in equations (9) and in equations (11) upon recognizing that there are n_f=16=L design harmonics k_o=k=0,1,. . .,15. There is no reason to use the negative harmonics since the spectrum is real and symmetric about the 0 frequency. Step 4.3 computes the passband peak-to-peak ripple as xripple and the stopband attenuation as xstop.

FIG. 5E 33 Step 5 calculates the weighted passband error metric w(pass)J(pass)=beta_pass, weighted stopband error metric w(stop)J(stop)=beta_stop, and weighted deadband error metric w(dead)J(dead)=beta_dead in equations (27) using the b_vector format for hn and the LS error matrices passband, stopband, deadband.

FIG. 5E,5F 34 Step 6 calculates the ISI and ACI LS error matrices, their metrics, and their error contributations to the SNR loss. Step 6.1 calculates the ISI LS error matrix w_matrix and the corresponding ISI metric J(ISI)=errM_isi with Matlab code which implements equations (25), and calculates the ISI residual error errV_isi that causes an ISI SNR loss. Step 6.2 calculates the ACI LS error matrix w_f_matrix and the corresponding ACI metric J(ACI)= errM_aci with Matlab code which implements equations (26), and calculates the ACI residual error errV_aci that causes a ACI SNR loss.

FIG. 5G 35 Step 7 calculates the weighted LS error metrics for ISI and ACI and the LS error matrix pw_t for the next iteration. Step 7.1 calculates the weighted ISI LS error metric w(ISI)J(ISI)=beta_isi and the weighed ACI LS error metric w(ACI)J(ACI)=beta_aci in equations (27) and the sum of the weighted LS error metrics J=errM_isi. Step 7.2 saves the weighted

LS error metrics and their sum J for each iteration. Step 7.3 updates the pw_t matrix for the next iteration.

FIG. 5G,5H 36 Step 8 calculates the SNR losses, stores these losses for each iteration, and completes the iteration loop. Step 8.1 computes the SNR loss in dB due to passband ripple as xloss_ripple, due to ISI as xloss_isi, due to ACI as xloss_aci, and the total SNR loss as xloss_total. Step 8.2 saves these SNR losses for each iteration and completes the iteration loop.

FIG. 5H,5I,5J,5K 37 Step 9 documents the Wavelet ψ frequency design harmonics and time response generated by the Matlab code for the Wavelet in FIG. ψ 6. Step 9.1 lists the Wavelet ψ frequency domain harmonic design coordinates $\psi_k(k)$=hw_eig(k) for harmonics k=0,1,…,15. Only the positive harmonics are listed since the frequency spectrum is real and symmetric about k=0. Step 9.2 lists the Wavelet ψ time response ψ(n)=hn(n) for n=0,1,2,. . . ,128 with the coordinate frame centered at the Wavelet peak. Only the positive digital samples are listed since the time response is real and symmetric about n=0.

FIG. 5K 38 Step 10 plots the total weighted LS error metric J in figure(1) and the individual weighted LS error metrics in figure(2) as functions of the iteration number. After about 6-7 iterations both the total and the individual weighted LS error metrics are observed to stabilize indicating that the selected number of iterations n_iteration=10 is sufficient for convergence.

FIG. 5L 39 Step 11 plots the Wavelet ψ frequency response in figure(3) which is the same as the frequency response in FIG. 6, plots the SNR losses due to passband ripple, ISI, ACI and the

total loss in figure(4), and plots the time response in figure(5) for use in the optimization of the LS metric weights in Step 1.3 as well as the scenario parameters in Step 1.1.

FIG. 5M 40 Step 12 documents the Matlab code to rescale the mother Wavelet ψ in FIG. 6 which was derived in the previous Matlab code in Steps 1-11, for new scale (dilation) p, time translation q, and frequency translation k parameters compared to the values p=0,q=0,k=0 for the mother Wavelet ψ. An example set of new parameters documented in the code is p=2, q=2, k=3 and with the assumption that the digital sampling rate remains fixed under the parameter change. A constant digital sample rate keeps the frequency band constant under the parameter change.

FIG. 5M 40 Step 12.1 defines the new Wavelet $\psi_{p,q,k}$ in equations (7),(18) resulting from these parameter changes, in equations (28)

New Wavelet scaled from mother Wavelet (28)

$$\psi_{p,q,k} = 2^{-p/2}\ \psi(2^{-p}n-qM)\ \exp(j2\pi\ k\ 2^{-p}n\ /ML)$$

Case 1 Scaled (dilated) sampling $2^{-p}n$ with fixed M subsamples (decimates) n by the factor 2^p

- Represent n as
  $n = n\_0 + 2^p\ n\_p$
  $n\_0 = 0,1,\ldots, 2^p-1$
  $n\_p = 0, +/-1, +/-2, \ldots$
- We observe $2^{-p}n = n\_p$
- This enables the new Wavelet to be written
  $\psi_{p,q,k} = 2^{-p/2}\ \psi(n\_p-qM)\ \exp(j2\pi\ k\ n\_p\ /ML)$

Case 2 Sampling n remains fixed and M is

rescaled by the factor 2^p

- New M is M_new

$$M\_new = 2^p M$$

- This enables the new Wavelet to be written

$$\psi_{p,q,k} = 2^{-p/2}\ \psi(n-qM\_new)\ \exp(j2\pi\ kn/M\_newL)$$

wherein the new Wavelet equation from (7),(18) is re-written using the frequency translation parameter k=r and the explicit frequency translation expression $\exp(j2\pi\ k\ 2^{-p}n\ /ML)$ which is recognized as the discrete Fourier transform kernal.

In Case 1 corresponding to the standard Wavelet assumption, the subsampling or decimation of n by the factor 2^p stretches the time between contiguous samples by the factor 2^p while keeping constant the number of samples M per Wavelet ψ sample interval, and therefore reduces the frequency band by the factor 2^p. The new n can be represented as the sum of the two fields n_0 = 0,1,. . ,(2^p -1) and n_p = 0, +/-1, +/-2, . . . Scaling or equivalently subsampling results in the new n equal to n_new = 2^-p n = n_p corresponding to a stretching of the sample interval by 2^p. The new Wavelet length N_new remains constant N_new=N using the n_p sampling.

In Case 2 which is the communications application being considered, the sampling is held constant corresponding to a constant frequency band assumption. The factor 2^-p is divided out in the Wavelet expression which is equivalent to the M being scaled to M_new=2^p M corresponding to increasing the number of samples per Wavelet interval by the factor 2^p. This means Case 2 corresponds to increasing the maximum number of channels by the factor 2^p over the same frequency band. The new Wavelet length is equal to N_new=M_newL+1 =1025 since M_new=2^pM=64.

FIG. 5M 40 Step 12.2 derives the new matrix transformation bw_matrix_new which maps the ψ frequency domain design eigenvectors into the one-sided ψ time response.

FIG. 5M,5N 40 Step 12.3 uses the matrix transformation bw_matrix_new to map the design eigenvector hw_eig into the new Wavelet in three steps. In the first step, the mother Wavelet is translated into the new baseband Wavelet $\psi_{p,q=0,k=0}$ = hn_0 which is the mother Wavelet scaled by p and with no time and frequency translations q=k=0. The second step translate this baseband Wavelet by qM_new to generate $\psi_{p,q,k=0}$ = hn_1. The last step translates this Wavelet $\psi_{p,q,k=0}$ = hn_1 in frequency to give the new Wavelet $\psi_{p,q,k}$ = hn_new.

FIG. 5N 40 Step 12.4 plots the Wavelet time responses for the mother Wavelet ψ=hn and the new Wavelet $\psi_{p,q,k}$ = hn_new in figure(6). Step 12.5 plots the Wavelet frequency responses for the mother Wavelet ψ=hn and the new Wavelet $\psi_{p,q,k}$ = hn_new in figure(7) versus the normalized frequency/(ψ=hn sample rate) and in figure (8) versus the normalized frequency/($\psi_{p,q,k}$ = hn_new sample rate).

FIG. 5O 41 Step 13 documents the functions used in the Matlab code. Step 13.1 is the code for the function pmn which computes the matrix for the J(BAND) in equations (23). FIG. 5P 41 Step 13.2 lists the code for the function freq_rsp which computes the Fourier transform of the input hn versus the normalized frequency/(Wavelet ψ=hn sample rate).

11. Wavelet applications

Applications of this new invention to both communications and radar include the use of the new Wavelet: 1) for communications applications which use the square-root raised cosine waveform (sq-root rc) which is extensively used for the third generation (3G) CDMA communications, 2) for communications applications which use the Gaussian minimum shift keying (GMSK) waveform for constant amplitude bandwidth efficient (BEM) applications, and 3) for synthetic aperture radar (SAR) and real aperture radar (RAR) applications.

FIG. 6 illustrates the communications applications to filters and waveforms currently using the square-root raised-cosine (sq-rt rc) waveforms with bandwidth expansion parameter $\alpha = 0.22$ to $\alpha = 0.40$. This notation means that for $\alpha = 0.22$ the spectral efficiency is (symbol rate/bandwidth) = $1/1+\alpha$ =1/1.22 = 0.82 = 82%. CDMA communications applications for sq-rt rc waveforms include current CDMA and 3G CDMA. FIG. 6 plots dc power spectral density or power spectrum (PSD) in units of dB 42 versus the frequency offset from dc expressed in units of the symbol rate 43 for the Wavelet waveform 44, the sq-rt r-c with α=0.22 45, and the sq-rt r-c with α=0.40 46.

12. Wavelets for minimum shift keying BEM

Constant amplitude BEM application of the new Wavelet waveform indicates that it can be used for communications applications of GMSK. The GMSK finds applications for transmitters which operate their HPA(s) amplifiers in a saturation mode in order to maximize their radiated power from

the HPA(s), and which require a BEM PSD to avoid excessive spreading of the transmitted power.

FIG. 7 plots the simulation data for the Wavelet waveform BEM 49 and the GMSK 50 as PSD in dB 47 versus the normalized frequency offset from dc expressed in units of the bit rate 48. Both Wavelet and GMSK use a a length parameter L=10 where L is the length of the phase pulse in units of phase pulse repetition rate.

13. Wavelets for synthetic aperture radar

Radar RAR and SAR applications of the Wavelet waveform indicate that it can be used for the chirp waveforms for wideband signal transmission, when combined with pseudo-random phase codes.

FIG. 8 plots the results of the simulation for the Wavelet waveform and an unweighted frequency chirp waveform. Plotted are the ambiguity function for the Wavelet waveform 51 and the unweighted frequency chirp waveform 52. The dc 2-dimensional radar ambiguity function 53 is plotted as a function of the frequency offset in units of fT_p 54 and the time offset in units of t/T_c 55 where T_p is the phase-coded radar pulse length or length of the phase code and T_c is the phase code chip length. The chip length is identical to the waveform repetition interval T_s so that $T_c = T_s$.

14. Preferred embodiments

Preferred embodiments in the previous description is provided to enable any person skilled in the art to make or use the present invention. The various modifications to these embodiments will be readily apparent to those skilled in the art, and the generic principles defined herein may be applied to other embodiments without the use of the inventive faculty. Thus, the present invention is not intended to be limited to the embodiments shown herein but is to be accorded the wider scope consistent with the principles and novel features disclosed herein.